BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W. WASHINGTON, D.C. 20024

SUBJECT: Digital Simulations of Random Crew Motion Disturbances
Case 620

DATE: December 31, 1968

FROM: N. I. Kirkendall

ABSTRACT

This memorandum describes a technique for digital simulation of crew motion disturbances that are described by random processes of specified power spectral density. The purpose of the simulation is to evaluate the performance of the ATM Pointing Control System under the action of random crew motion disturbances.

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MEMORANDUM FOR FILE

In a massive spacecraft, such as we have in the AAP cluster, crew motion disturbances may be treated as forces and moments applied to specific points. Heretofore these forces and moments have been described by deterministic waveforms.(1)(2) These waveforms are suitable for studying the transient response of a pointing control system such as on the ATM. They are not appropriate, however, for determining the long term system performance by a criterion such as the mean-square pointing error. For this reason MSFC is now developing so called statistical models of crew motion. These are just the power spectral densities of forces and moments for classes of crew motion.

In order to make use of these models in the Bellcomm simulation of pointing control systems,(2) a technique is needed for the digital simulation of a random process with a specified rational power spectral density. This memorandum describes such a technique. The order of our discussion will be:

(1) Finding the transfer function of a linear filter which has the property that its output has a specified power spectral density when its input is a specified random process.

(2) Finding the differential equation representation of the filter.

(3) Simulating the solution of the differential equation when white noise is the input(3).

(4) Testing the simulation(4).

(5) Example.

1. TRANSFER FUNCTION OF A LINEAR FILTER

Before going into detail on the generation of a random process, we want to point out that it is always possible to find the transfer function of a linear filter which produces a random process with a given rational power spectral density when driven by a specified random process.

If one is given a real polynomial $S(\omega)$ such that there exists a real polynomical $R(\omega)$, with the property that $S(\omega) = R(j\omega)\ R(-j\omega)$, then it will be possible to uniquely determine R by multiplying $R(j\omega)\ R(-j\omega)$ and equating coefficients of ω^k for $k=0, \ldots n$, where n is the order of $S(\omega)$. This process yields n+1 equations in n+1 unknowns, which have unique solutions. Therefore, since a rational power spectral density function $S(\omega)$ is a ratio of two such polynomials $S_1(\omega)/S_2(\omega)$, the transfer function of a linear filter $P(j\omega)/Q(j\omega)$ can be uniquely determined.

2. DIFFERENTIAL EQUATION REPRESENTATION OF FILTER

For the remainder of this discussion it will be assumed that we want to simulate a random process using a filter with a known rational transfer function. Therefore if $\eta(t)$ is some random input signal we have

$$\text{Input} \rightarrow \qquad \text{Filter} \qquad \rightarrow \text{Output}$$

$$\eta(t) \rightarrow \frac{P(S)}{Q(S)} = \frac{b_o + b_1 S + \ldots + b_K S^k}{a_o + a_1 S + \ldots + a_n S^n} \rightarrow y(t)$$

where $y(t)$ is the desired random process, and $k<n$. $P(S)/Q(S)$ represents the Laplace transform of the impulsive response of the filter. We would now like to find the linear differential equation which also represents this filter.

We may write

$$y(S) = \frac{\eta(S)}{a_o + a_1 S + \ldots + a_n S^n}\,(b_o + b_1 S + \ldots + b_k S^k)$$

Letting

$$X_1(S) = \frac{\eta(S)}{a_o/a_n + a_1/a_n\ S + \ldots + a_{n-1}/a_n S^{n-1} + S^n}$$

we get

$$y(S) = 1/a_n\,(b_o + b_1 S + \ldots + b_k S^k).\ X_1(S)$$

By rearranging terms in the definition of $X_1(S)$ we have

$$\frac{a_o}{a_n} X_1(S) + \frac{a_1}{a_n} SX_1(S) + .. + S^n X_1(S) = \eta(S).$$

Taking the inverse Laplace transform yields*

$$\frac{a_o}{a_n} X_1(t) + \frac{a_1}{a_n} X_1^{(1)}(t) + \ldots + \frac{a_{n-1}}{a_n} X_1^{(n-1)}(t)$$

$$+ X_1^{(n)}(t) = \eta(t)$$

and letting

$$\begin{aligned} X_2 &= \dot{X}_1 &&= X_1^{(1)} \\ &\vdots && \vdots \\ X_n &= \dot{X}_{n-1} &&= X_1^{(n-1)} \end{aligned}$$

we get

$$\dot{X}_n = \eta(t) - \frac{a_o}{a_n} X_1 - \ldots - \frac{a_{n-1}}{a_n} X_n.$$

* $x_1^n(t) = \frac{d^n x_1}{dt^n}$

Therefore writing in matrix notation we have

$$\dot{X} = A X + G \eta \qquad \text{and} \qquad y = M^T X$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \vdots \\ -\frac{a_o}{a_n} & & \cdots & & -\frac{a_{n-1}}{a_n} \end{pmatrix} \qquad G = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \qquad M = \begin{pmatrix} -\frac{b_o}{a_n} \\ \vdots \\ -\frac{b_k}{a_n} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The equation for y comes from the definition for X and the fact that

$$y(S) = X_1(S) \quad \frac{1}{a_n} \quad (b_o + b_1 S + .. + b_k S^k)$$

The above argument illustrates how to find the linear differential equation that represents a given rational transfer function. Therefore, if we can find a random process which is a solution of the differential equation $\dot{X} = AX + G\eta$ we also have the random process y, since $y = M^T X$.

3. SIMULATION OF THE RANDOM PROCESS(3)

It is known that

(1) $$X(t) = \phi(t,t_o)X(t_o) + \int_{t_o}^{t} \phi(t,\tau)G(\tau)\eta(\tau)d\tau$$

is the solution to $\dot{X}(t) = AX(t) + G(t)\ \eta(t)$, where t_o is an arbitrary starting time and $\phi(t,t_o)$ is the state transition matrix. $\phi(t,t_o)$ specifies the evolution of the system from time t_o to time t, and has the following properties:

a. $\phi(t_o,t_o) = I$ (identity matrix)

b. $\frac{d}{dt}(\phi(t,t_o)) = A \cdot \phi(t,t_o)$

Since the eigenvalues of A have negative real parts* we know that for any finite t, $\phi(t, -\infty) = 0$. Also, using relations (a) and (b) we see that

$$\phi(t_2,t_1) = \mathcal{L}^{-1}(SI-A]^{-1}_{t=t_2-t_1}$$

Thus $|SI-A| = 0$ is the characteristic equation of the filter.

Now in expression (1), $\phi(t,t_o)$ and G are defined. Expression (1) indicates that the value of X for any $t>t_o$ depends on its value at t_o. Therefore we would like to find a sample of the process X at t_o. For subsequent values of t, we will need to find samples of the process

$$Z(t,t_o) = \int_{t_o}^{t} \phi(t,\tau)G(\tau)n(\tau)d\tau$$

Then by combining these samples as prescribed in (1), we would obtain a sample of the random process X for any value of t.

Before proceeding with the discussion of the random process, it is useful to point out that it is possible to generate a random vector having any specified covariance matrix. We will later use this fact in finding a sample of our random process. If we are given any covariance matrix V (which is nxn and symmetric), it is possible to find a lower triangular matrix C such that $V = CC^T$. If

$$V = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_{22} & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \sigma_{1n} & \cdots & & \sigma_{nn} \end{pmatrix} \text{ and } C = \begin{pmatrix} c_{11} & 0 & \cdots & 0 \\ c_{12} & c_{22} & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & 0 \\ \cdot & & & \\ c_{1n} & \cdots & & c_{nn} \end{pmatrix}$$

*A linear passive filter is assumed

then by multiplying CC^T and setting the ij^{th} element of CC^T equal to the ij^{th} element of V, we obtain n! equations in n! unknowns. Therefore if we are given a covariance matrix V, and we find the above matrix C, it is possible to find a random vector which has covariance V. If $\omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}$ represents a sample of n numbers drawn independantly from a distribution which is normal (0,1), then it is known that the distribution of ω is normal (0,I). If we define $Z = C\omega$, Then

$$\text{Cov } (Z) = E(ZZ^T) = E(C\omega\omega^T C^T) = CE(\omega\omega^T)C^T = CIC^T = V$$

and the distribution of Z is normal (0,V).

Now, returning to our main problem, in order to generate a sample of $X(t_o)$ we will find cov $(X(t_o))$, and as above generate a random sample with that covariance. We will also find cov $(Z(t,t_o))$ and generate another random sample with that covariance. We will then combine them as in (1) to produce a vector X(t).

$$\text{Cov } (X(t_o)) = E(X(t_o)X^T(t_o))$$

$$= E\left(\int_{-\infty}^{t_o} \phi(t_o,\tau)G(\tau)\,\eta(\tau)d\tau \left(\int_{-\infty}^{t_o} \phi(t_o,\rho)G(\rho)\,\eta(\rho)d\rho\right)^T\right)^*$$

$$= E\left(\int_{-\infty}^{t_o}\int_{-\infty}^{t_o} \phi(t_o,\tau)G(\tau)\eta(\tau)\eta^T(\rho)G^T(\rho)\phi^T(t_o,\rho)d\tau d\rho\right)$$

$$= \int_{-\infty}^{t_o}\int_{-\infty}^{t_o} \phi(t_o,\tau)G(\tau)\left[E(\eta(\tau)\eta^T(\rho))\right]G^T(\rho)\phi^T(t_o,\rho)d\tau d\rho$$

*Follows from equation (1) since $\phi(t_o,-\infty) = 0$.

Up to now we have said nothing about the random process $\eta(t)$. It is observed at this point that if one wants only a starting point for a system with random input the above equation is valid. For our purposes though we will assume that $\eta(t)$ is white noise, $E(\eta(t)) = 0$, and $E(\eta(t), \eta(\rho)) = \delta(t-\rho)$. This assumption reduces the above integral to

$$\operatorname{cov}(X(t_o)) = \int_{-\infty}^{t_o} \phi(t_o,\tau) G G^T \phi^T(t_o,\tau) d\tau$$

In a similar manner we find

$$\operatorname{Cov}(Z(t,t_o)) = E\left(\int_{t_o}^{t}\int_{t_o}^{t} \phi(t,\rho) G(\rho)\, \eta(\rho)\, \eta^T(\tau) G^T(\tau) \phi^T(t,\tau) d\tau d\rho\right)$$

$$= \int_{t_o}^{t} \phi(t,\tau) G G^T \phi^T(t,\tau) d\tau$$

Thus a sample $X(t_o)$ is found, having its covariance given by $\operatorname{Cov}(X(t_o))$, and for any time t, a sample $Z(t,t_o)$ is found having covariance $\operatorname{Cov}(Z(t,t_o))$. Then $X(t)$ given by

$$X(t) = \phi(t,t_o) X(t_o) + Z(t,t_o)$$

is a sample from a process which is a solution to the differential equation

$$\dot{X} = AX + G\eta$$

A recursive formula, for $t_o = 0$, is

$$X(nT) = \phi(nT,(n-1)T) X((n-1)T) + Z(nT,(n-1)T) \quad .$$

The autocorrelation function of $y(t)$ is most easily found by taking the inverse Fourier transform of $S(\omega)$.

4. TESTING THE SIMULATION

In order to check the simulation of such a process, there are several sample statistics which will indicate whether the process being generated is indeed what we think it is. These sample statistics and their distributions are derived and discussed by J. L. Strand in Ref. 4. The sample statistics which seem most useful are the sample mean and the sample autocorrelation matrix

$$\bar{\mu} = \frac{1}{N+1} \sum_{i=0}^{N} Y(iT),$$

$$\bar{\rho}(kT) = \frac{1}{N-k+1} \sum_{v=o}^{N-k} Y(vT)Y^T((v+k)T) \quad .$$

where Y is the output vector of the process being generated.

The sample mean $\bar{\mu}$ has a normal distribution with $\{E(\bar{\mu})=0\}$ and variance

$$V(\bar{\mu}) = \frac{1}{N+1}\left[\rho(0) + 2 \sum_{i=1}^{N} (i-\frac{i}{N+1})\rho(iT)\right]$$

where $\rho(iT)$ represents the autocorrelation matrix of Y for $t = iT$, $i = 0, 1, \ldots N$.

The sample autocorrelation $\bar{\rho}(kT)$ is not normal but for sufficiently large N*, it may be assumed to be so. For any N, $E(\bar{\rho}(kT)) = \rho(kT)$, and

$$V(\bar{\rho}_{ij}(kT)) = \frac{1}{N-k+1}\Big[\rho_{ii}(0)\rho_{jj}(0) + \rho^2{}_{ij}(kT)$$

$$+ 2 \sum_{v=1}^{N-k} (1 - \frac{v}{N-k+1})(\rho_{ii}(vT)\rho_{jj}(vT)$$

$$+ \rho_{ij}((v+k)T)\rho_{ij}((v-k)T)\Big]$$

where ρ_{ij} represents the ij^{th} element of the autocorrelation matrix ρ.

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*Central limit theorem

Therefore, for sufficiently large N, it is possible to use the standard deviation of the statistics $\bar{\mu}$ and $\bar{\rho}(kT)$ as a yardstick to see how close their observed values are to their desired values. N must be large since this type of test is valid only for normal distributions.

5. EXAMPLE

For crew motion, it was initially assumed that

$$\frac{P(S)}{Q(S)} = \frac{kS^2}{(S\tau_1+1)^2(S\tau_2+1)}$$

The associated differential equation is

$$\dot{X} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \dfrac{-1}{\tau_1^{\,2}\tau_2} & \dfrac{-(\tau_2+2\tau_1)}{\tau_1^{\,2}\tau_2} & \dfrac{-(2\tau_2+\tau_1)}{\tau_1\tau_2} \end{pmatrix} X + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \eta$$

$$y = \left(0 \quad 0 \quad \frac{k}{\tau_1^{\,2}\tau_2}\right) X$$

The transition matrix is given by

$$\phi_{ij}(t_2,t_1) = (A_{ij} + (t_2-t_1)B_{ij}) \exp\left(-\frac{t_2-t_1}{\tau_1}\right) + C_{ij} \exp\left(-\frac{t_2-t_1}{\tau_2}\right)$$

where A_{ij}, B_{ij}, C_{ij} are constants which depend only on τ_1 and τ_2. After generating the solution $X(t)$ (for $k=1, \tau_1=.318$, $\tau_2=.0398$, $T=.01$) and computing the test statistics $\bar{\mu}, \bar{\rho}(T)$, and $\bar{\rho}(0)$ and their distributions for $N=1000$, it was determined that the simulation was acceptable. The results of the tests appear in the following table.

Error in such a simulation, assuming all programming and hand calculations are correct, should be due only to deficiencies in the random number generator.

Nancy I. Kirkendall

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N. I. Kirkendall

Attachments
  References
  Trials

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2. Smith, P. G., "A Mathematical Model for Simulation of the Apollo Telescope Pointing Control System", Bellcomm Technical Report TR-68-620-1.

3. Lionel J. Skidmore, "Probability and Random Processes," Presented in a short course entitled: "Space Navigation and Guidance," University of California Extension, Los Angeles, 16 October 1967.

4. Strand, John L., "Simulating the Solution of a Stochastic Differential Equation," Technical Memorandum, TM-68-1033-7 December 31, 1968.

TRIALS

	1	2	3	4	5
Sample Mean $\bar{\mu}$					
$E(\bar{\mu}) = 0$					
$(V(\bar{\mu}))^{1/2} = 1.68 \times 10^{-3}$					
Observed Values	1.38×10^{-2}	-3.34×10^{-3}	-1.40×10^{-3}	3.88×10^{-4}	2.21×10^{-3}
Autocorrelation $\bar{\rho}(.01)$					
$E(\bar{\rho}(.01)) = 1.23 \times 10^{-2}$					
$(V(\bar{\rho}(.01)))^{1/2} = .134 \times 10^{-2}$					
Observed Values	1.31×10^{-2}	1.32×10^{-2}	1.09×10^{-2}	1.48×10^{-2}	1.13×10^{-2}
Variance $\bar{\rho}(o)$					
$E(\bar{\rho}(o)) = 1.67 \times 10^{-2}$					
$(V(\bar{\rho}(o)))^{1/2} = 2.36 \times 10^{-2}$					
Observed Values	1.74×10^{-2}	1.75×10^{-2}	1.54×10^{-2}	1.95×10^{-2}	1.57×10^{-2}

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